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XXVI.—On the Rotatory Motion of the Heavenly Bodies. By the Rev. W. G. Penny, M. A., Professor of Mathematics in the Catholic University of Ireland, and late Mathematical Scholar in the University of Oxford.

[Read February 24, 1868.]

1. The object of the present Paper is, in the first place, to ascertain whether the various disturbing forces which act upon the heavenly bodies produce any permanent effect upon their rotation; and, secondly, supposing such an effect to exist in general, to ascertain under what circumstances it will cease to do so—that is to say, what are the conditions under which they would rotate permanently without any but periodic changes.

The inquiry may be of some interest, for two reasons:—First, it has been ascertained that the observed acceleration in the motion of the moon has been only partially accounted for by the diminution of the excentricity of the earth's orbit, to which cause part of it, though not much more than half, is undoubtedly due; that is, if we calculate what ought to have been the angular distance of the sun and moon at the time of an ancient eclipse—say 2500years ago—it is found that, after making allowance for the acceleration produced by the cause mentioned, that their angular distance so calculated does not agree with what it was actually observed to have been. Now, such an error might be produced either by an error in the supposed velocity of the moon, or of that of the sun, i. e. of the earth ; but both these have been carefully examined, and found to be inadequate to explain the phenomena. There is, however, a third cause which would give rise to the same discrepancy between theory and observation—namely, *an error in the measure of time*—just as an error in a ship's longitude at sea might be caused by an error in the rate of the chronometer. Just so an error in the angular distance of the sun and moon a certain number of years ago might be caused by an error of any one of the three elements which enter into it—the length of the year, the length of the month, and the length of the day; and so it is evident that, if the length of the day has undergone any sensible alteration since the date of the eclipse above spoken of, the real length of time that has elapsed since then must be different to what would be supposed if the length of the day had remained invariable; and thus the relative positions of the sun and moon would also be different. Accordingly, it was suggested by M. Delaunay that possibly the length of the day—or, in other words, the velocity of the earth's rotation—had varied.

In the next place, the moon, and it is said satellites generally, turn always the same face towards their primaries; that is, they rotate about their axes in the same time that they perform a revolution about their primary. And it has been asked whether there is any cause for this; in other words, whether the body about which they revolve exerts any influence upon them which would affect the rotatory motion, so as to make it coincide in period with that in the orbit, supposing that there had ever been a time when they did not do so.

The general results arrived at may be thus stated:—

(1). That in all bodies which are perfectly symmetrical with respect to the three planes containing their principal axes of rotation, such as an ellipsoid (not one of revolution), there is no permanent change produced; nor in any in which the moments of inertia about the two principal axes which are perpendicular to that about which it is revolving are equal; I think there is a permanent change whenever the three moments are unequal, though the body may be perfectly symmetrical with respect to each axis;

(2). That in other cases there is in general a permanent change produced;

(3). That in the case of the earth disturbed by the lunar moon, the change produced will be so very small as to account for a very minute fraction of the whole amount required to explain the phenomena above alluded to;

(4). That one condition under which there will be no permanent variation, is when the time of rotation nearly or exactly coincides with that in the orbit; but this is only one out of several other such relations as might exist; just as there are always several positions in which a body might remain in statical equilibrium; and that in some cases, though not in all, the forces are such as to produce the relation above spoken of; and, lastly,

(5). That the effects are enormously more rapid in the case of a satellite described by its primary than *vice versâ.*

I have supposed, in treating the question, only one disturbing force to be acting upon the body, and also that its orbit is a fixed plane; neither of which, especially for the earth or moon, is strictly the case, but will be sufficiently near the truth for the present purpose; also, I have supposed the body to be entirely solid, instead of being partially covered with a thin layer of fluid. Mr. Airy, however, is said to have examined the effect of the tidal wave, which it was supposed might by its position, &c., produce some retardation upon the earth's motion, and has found it to be insensible.

Result (4.) has been spoken of as only an approximate one—indeed, to pretend to extract anything more out of differential equations which can only be solved by successive approximations, as is the case in the present instance, would seem almost to amount to a contradiction of terms.

2. The differential equations of motion are

$$\left.\begin{aligned}
\frac{d\omega_1}{dt} + \frac{C-B}{A}\,\omega_3\omega_2 &= \frac{1}{A}\,\Sigma m\,\frac{zy_1 - yz_1}{\left((x-x_1)^2+(y-y_1)^2+(z-z_1)^2\right)^{\frac{3}{2}}}\\
\frac{d\omega_2}{dt} + \frac{A-C}{B}\,\omega_3\omega_1 &= \frac{1}{B}\,\Sigma m\,\frac{xz_1 - zx_1}{\left\{\overline{x-x_1}^2+\overline{y-y_1}^2+\overline{z-z_1}^2\right\}^{\frac{3}{2}}}\\
\frac{d\omega_1}{dt} + \frac{B-A}{C}\,\omega_1\omega_2 &= \frac{1}{C}\,\Sigma m\,\frac{yx_1 - xy_1}{\left\{(\overline{x-x_1}^2+\overline{y-y}^2+\overline{z-z_1}^2\right\}^{\frac{3}{2}}}
\end{aligned}\right\}\ A.$$

The solution of these equations will give the values of ω_1, ω_2, and ω_3, and these being known, the value of $\omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}$, or the value of the velocity about the actual or instantaneous axes of rotation is known also; and if any one of the quantities ω_1, ω_2, or ω_3, contain any term which increases with the time, and is not periodic, there will be a permanent change. The body is supposed to be rotating very nearly about the axis of z, about which ω_3 is the velocity, and where ω_1, ω_2, ω_3, are the angular velocities about the principal axes, A, B, C, the moments of inertia about the same; also x_1, y_1, z_1, are the co-ordinates of any particle m referred to the principal axes, the origin being at the centre of gravity of the disturbed body, and x, y, z, are the co-ordinates of the disturbed body, the plane of x, y being that of the orbit, the intersection of the planes of x_1, y_1, and x, y, the axis of x, and the origin as before.

Let i be the angle between the planes of x, y, and x_1, y_1; θ the longitude of the disturbed body measured from the axis of x on the plane of the orbit, ϕ the right ascension of the axis of x_1 measured on the plane of x_1, y, that is, the angular distance of the axis of x_1 from that of x; r the distance of the disturbing body from the centre of the disturbed, and r_1 the distance of any particle m from the origin. Then we shall have by spherical trigonometry,

$$\frac{x}{r} = \frac{1 + \cos \iota}{2} \cos (\phi - \theta) + \frac{1 - \cos \iota}{2} \cos (\phi + \theta)$$

$$\frac{y}{r} = - \frac{1 + \cos \iota}{2} \sin (\phi - \theta) - \frac{1 - \cos \iota}{2} \sin (\phi + \theta)$$

$$z = - \sin \iota \sin \theta.$$

Equations which are usually given in the form

$$\frac{x}{r} = \sin \theta \cos \phi + \cos \iota \cos \theta \sin \phi, \text{ \&c.}$$

But the form given above will be much the most convenient for the present purpose. The equations (A) can only be solved by successive approximation. The first approximation will be when the right hand member is 0, that is, when there is no disturbing force, or when the disturbed body is spherical. The next will be when the bodies are supposed to be spheroids of revolution. This very nearly represents the case of the heavenly bodies; but inasmuch as they have a variety of irregularities both of form and density, will not *accurately* do so; and it becomes therefore necessary to examine what will be the general effect of the said inequalities of surface; and specially to see whether there will be any permanent alteration in the velocity of rotation arising from them. Supposing such to exist, it is manifest that in consequence of the bodies being so nearly spherical, it will take place very slowly; but the ultimate amount of alteration will be none the less than if the inequalities which produced it were more considerable; only it will take more time to arrive at a permanent state.

To calculate, then, the effect of the disturbing force, we must develop the right-hand members of equations (A). To begin with the first. The quantity to be developed may be put in the form

$$\Sigma m\,(zy_1 - yz_1)\,\{(x^2+y^2+z^2) - 2\,(xx_1+yy_1+zz_1) + x_1^2+y_1^2+z_1^2\}^{-\frac{3}{2}}$$

or

$$\frac{1}{r^3}\,\Sigma m\,(zy_1 - yz_1)\Big(1 - 2\,\frac{xx_1+yy_1+zz_1}{r^2} + \frac{r_1^2}{r^2}\Big)^{-\frac{3}{2}}$$

$$= \frac{1}{r^3}\,\Sigma m\,(zy_1 - yz_1)\Big(1 + 3\,\frac{xx_1+yy_1+zz_1}{r^2} - \frac{3}{2}\,\frac{r_1^2}{r^2} + \frac{15}{2}\,\frac{(xx_1+yy_1+zz_1)^2}{r^4} +, \&\text{c.}\Big) \qquad (\beta)$$

$$= \frac{1}{r^3}\Sigma m\,(zy_1 - yz_1) + \frac{3}{r^3}\,\Sigma m\,(zy_1 - yz_1)\,(xx_1+yy_1+zz_1)$$

$$= \frac{1}{r^3}\,(z\Sigma\,(my_1) - y\Sigma\,(mz_1) + \frac{3}{r^5}\,(zx\,\Sigma\,(mx_1y_1) + zy\,\Sigma m\,(y_1^2 - z_1^2) + (z^2 - y^2)\,\Sigma m\,(\,y_1z_1 - xy\,\Sigma(m\,z_1x_1)$$

rejecting for the present the further terms in the development.

By the property of the centre of gravity and the principal axes, the terms $\Sigma(m\,y_1)$, &c., and also $\Sigma(m\,x_1y_1)$, &c., vanish; and it is reduced to

$$\frac{3}{r^5}\,zy\,\Sigma m\,(y_1^2 - z_1^2), \quad \text{or} \quad \frac{3}{r^5}\,zy\,\Sigma m\,(y_1^2 + x_1^2 - (z_1^2 + x_1^2))$$

or

$$\frac{3}{r^5}\,zy\,(B - C),$$

hence the first of equations (A) becomes

$$\frac{d\omega_1}{dt} + \frac{B - B}{A}\,\omega_3\omega_2 = \frac{3\mu}{r^5}\,zy\,\frac{C - B}{A}$$

similarly

$$\frac{d\omega_2}{dt} + \frac{A - C}{B}\,\omega_3\omega_1 = \frac{3\mu}{r^5}\,zx\,\frac{A - C}{B}$$

$$\frac{d\omega_3}{dt} + \frac{B - A}{C}\,\omega_1\omega_2 + \frac{3\mu}{r_5}\,xy\,\frac{B - A}{C}$$

and if in these we substitute the values of xy and z, given above, they will become

$$\frac{d\omega_1}{dt} + \frac{C - B}{A}\,\omega_3\omega_2 = \frac{3}{2}\,\frac{\mu}{r^3}\,\frac{C - B}{A}\,\sin\iota\Big(\frac{1 + \cos\iota}{2}\,\cos(\phi - 2\theta)\;\cos\iota\,\cos\phi - \frac{1 - \cos\iota}{2}\,\cos(\phi + 2\theta)\Big)$$

$$\frac{d\omega_2}{dt} + \frac{A - C}{B} \omega_3\omega_1 = \frac{3}{2} \frac{\mu}{r^3} \frac{A - C}{B} \sin \iota \left(\frac{\cos \iota + 1}{2} \sin (\phi - 2\theta) - \right.$$
$$\left. \cos \iota \sin \phi + \frac{\cos \iota - 1}{2} \sin (\phi + 2\theta) \right\}$$

$$\frac{d\omega_3}{dt} + \frac{B - A}{C} \omega_1\omega_2 = - \frac{3}{2} \frac{\mu}{r^3} \frac{B - A}{C} \left\{ \frac{\overline{1 + \cos \iota}^2}{2} \sin (2\phi - 2\theta) \right.$$
$$\left. + \tfrac{1}{2} \sin^2\iota \sin 2\phi + \frac{\overline{1 - \cos \iota}^2}{2} \sin (2\phi + 2\theta \right\}$$

In the first of these three there are three terms on the right-hand side, each of which being integrated will give a term in ω_1 of the general form $H \sin (\chi)$; and corresponding to this there will also be a term in ω_2 of the form $K \cos (\chi)$. Now, if these two are substituted for ω_1 and ω_2 in the function $\frac{B - A}{C} \omega_1\omega_2$ which occurs on the left side of the third of the above equations, it is evident that they will only produce a periodic term; but this is because one of them is a sine, and the other a cosine. If, however, they had both of them been sines, or both cosines, the case would have been very different, and the multiplication of them together would have produced a constant term. What we have to do, therefore, is to see whether the further development of the disturbing function will produce any such terms. And it is very readily that it does produce a considerable number of them, corresponding to different combinations of ϕ and θ of the same kind as those in the equations last formed. Those which I shall select for examination at present are those which have $\phi - \theta$ for their argument; so that we must develop the disturbing function so as to include all terms of the form $\sin \overline{\phi - \theta}$ or $\cos \overline{\phi - \theta}$ wherever they occur in the first two equations; that is, those for ω_1 and ω_2.

Let us resume, therefore, the two last terms in equation (β) which had been rejected, and which will contain all the terms of lowest dimensions of the form required, we shall then have for this part of the function on the right side of the equation for ω_1

$$\frac{1}{A} \frac{\mu}{r^3} \left\{ - \frac{3}{2} \Sigma m (zy_1 - yz_1) \frac{r_1^2}{r^2} + \frac{15}{2} \Sigma m (zy_1 - yz_1) \frac{(xx_1 + yy_1 + zz_1)^2}{r^4} \right.$$

substituting the values of z, y, &c., in this, and retaining only terms of the form $\sin \overline{\phi - \theta}$, $\cos \overline{\phi - \theta}$, the first part of this expression will give

$$- \frac{1}{A} \frac{3}{2} \frac{\mu}{r^4} \Sigma (m r_1^2 z_1) \frac{1 + \cos \iota}{2} \sin (\phi - \theta)$$

for shortness, let

$$\frac{1 + \cos \iota}{2} = \alpha \text{ and } \frac{1 - \cos \iota}{2} = \beta$$

also let $\Sigma\,(m' r_1{}^2 z_1)$, &c., be denoted by $\overline{r_1{}^2 z_1}$, and similar expressions for the others.
Also the latter part will give

$$-\frac{1}{A}\frac{15}{2}\frac{\mu}{r}\Sigma m\,(zy_1 - yz_1)\,(x^2x_1{}^2 + y^2y_1{}^2 + z^2z_1{}^2 + 2xy\,x_1y_1 + 2xz\,x_1z_1 + 2yzy_1z_1)$$

Multiplying these two factors together, and retaining all such terms as are either of one dimension x and y, or of three dimensions, which are the only ones which will produce terms of the form required, we have

$$-\frac{1}{A}\frac{15}{2}\frac{\mu}{r}\left(x^2y_1\,\overline{x_1{}^2z} + (y^3 - 2z^2y)\,\overline{y_1{}^2z_1} + z^2y\,\overline{z_1{}^3} + 2(xy^2 - xz^2)\,\overline{x_1y_1z_1}\right)$$

Now,

$$x = a\cos\overline{\phi - \theta} + \beta\cos\overline{\phi + \theta}, \quad \text{and} \quad y = -a\sin\overline{\phi - \theta} - \beta\sin\overline{\phi + \theta}$$

$$\therefore\ x^2 = \tfrac{1}{2}(a^2 + \beta^2) + \tfrac{1}{2}a^2\cos\overline{2\phi - 2\theta} + a\beta\cos 2\varphi + a\beta\cos 2\theta$$
$$+ \tfrac{1}{2}\beta^2\cos\overline{2\phi + 2\theta}$$

Multiplying this by the value of y, and retaining terms whose argument is $\overline{\phi - \theta}$,

$$x^2y = (-\tfrac{1}{2}(a^3 + a\beta^2) + \tfrac{1}{4}a^3 + \tfrac{1}{2}a\beta^2 - \tfrac{1}{2}a\beta^2)\sin\overline{\phi - \theta} = -\tfrac{1}{4}(a^3 + 2\beta^2a)\sin\overline{\phi - \theta}$$

which is the coefficient of $\overline{x_1{}^2z_1}$ above.
Also in like manner

$$y^3 = -\tfrac{3}{4}(a^3 + 2\beta^2a)\sin\overline{\phi - \theta}$$

and

$$z^2y = -\frac{\sin^2\iota}{2}(1 - \cos 2\theta)(\alpha\sin\overline{\phi - \theta} + \beta\sin\overline{\phi + \theta})$$
$$= -\tfrac{1}{4}\sin^2\iota\,(2a - \beta)\sin\overline{\phi - \theta}.$$

Thus the coefficient of $\overline{y_1{}^2z_1}$ becomes

$$-(\tfrac{3}{4}(a^3 + 2\beta^2a) - \sin^2\iota\,)\,4a - 2\beta))$$

and that of

$$\overline{z_1{}^3}\text{ is } -\tfrac{1}{4}\sin^2\iota\,\overline{2a - \beta}\sin\overline{\phi - \theta},$$

also

$$y^2 = \tfrac{1}{2}(a^2 + \beta^2) - \tfrac{1}{2}a^2\cos\overline{2\phi - 2\theta} + \tfrac{1}{2}a\beta\cos 2\theta - \tfrac{1}{2}a\beta\cos 2\phi$$
$$- \tfrac{1}{2}\beta^2\cos\overline{2\phi + 2\theta}$$

$$\therefore\ 2y^2 = \tfrac{1}{4}(a^3 + 2a\beta^2)\cos\overline{\phi - \theta}$$

and

$$xz^2 = \tfrac{1}{4}\sin^2\iota\,\overline{2a - \beta}\cos\overline{\phi - \theta}$$

so that the coefficient of

$$\overline{x_1 y_1 z_1} \text{ is } \tfrac{1}{2}(\alpha^3 + 2\alpha\beta^2 - \overline{2\alpha - \beta}) \cos \overline{\phi - \theta}$$

making these substitutions, we have

$$+\frac{1}{A}\frac{15}{8}\frac{\mu}{r^4}\left\{\alpha^3 + 2\beta^2\alpha) \overline{x_1^2 z_1} + (3(\alpha^3 + 2\beta^2\alpha) - \overline{4\alpha - 2\beta} \sin^2\iota) \overline{y_1^2 z}\right.$$

$$\left. + \sin^2\iota\, (2\alpha - \beta) \overline{z_1^3}\right\} \sin(\phi - \theta)$$

$$-\frac{1}{A}\frac{15}{4}\frac{\mu}{r^4}\; (\alpha^3 + 2\alpha\beta^2 - \overline{2\alpha - \beta} \sin^2\iota) \overline{x_1 y_1 z_1} \cos \overline{\phi - \theta}$$

and the whole coefficient of $\sin \overline{\phi - \theta}$ will be the quantity just found, minus the quantity first found above, that is, it will be

$$\frac{\mu}{r^4}\frac{1}{A}\left\{\frac{15}{8}\left\{(\alpha^3 + 2\beta^2\alpha) \overline{x_1^2 z_1} + (3(\alpha^3 + 2\beta^2\alpha) \sin^2\iota\, (4\alpha - 2\beta)) \overline{y_1^2 z}\right.\right.$$

$$\left.\left. + \sin^2\iota\, (2\alpha - \beta) \overline{z_1^3}\right\} - \frac{3}{2} \alpha\, \overline{r_1^2 z_1}\right\}$$

call this, for shortness, $\frac{1}{A} M_y$, and the coefficient of $\cos \overline{\phi - \theta}$, just given, $-\frac{1}{A} N$. Then the equation for ω_1 will become

$$\frac{d\omega_1}{dt} + \frac{C - B}{A} \omega_3\omega_2 = \frac{1}{A} M_y \sin \overline{\phi - \theta} - \frac{1}{A} N \cos \overline{\phi - \theta},$$

and in like manner

$$\frac{d\omega_2}{dt} + \frac{A - C}{B} \omega_3\omega_1 = -\frac{1}{B} N \sin \overline{\phi - \theta} + \frac{1}{B} M_x \cos \overline{\phi - \theta}$$

where M_x is what M_y becomes when x_1 and y_1 are interchanged. In order that these may be integrated, it will be necessary to express ϕ and θ in terms of t; but before doing so it may be well to make a remark upon the quantities $\Sigma(m\, x_1^2 z_1)$, &c., which occur continually. We may always choose the axes so as to make

$$\Sigma\,(m x_1) = 0, \quad \Sigma\,(m x_1 y_1) = 0, \text{ \&c.};$$

but not so as to satisfy any further conditions, such as

$$\Sigma\,(m x_1^2 y_1), \quad \Sigma\,(m x_1^2 z_1 = 0.$$

But there is one case in which a certain class of these quantities will always vanish. Whenever the body is perfectly symmetrical in term and density with respect to the planes which contain its principal axes, it is manifest that all quantities of the form $\Sigma\,(m\,x_1{}^p z_1{}^q)$, &c., when either p or q are odd numbers, will vanish. Thus, suppose q to be an odd number, then any particle m having z_1 for one of its co-ordinates will always be accompanied by another having $-z_1$ for one of its co-ordinates; hence the sum of these will vanish. Now, these are the sort of quantities, both in the example which I have chosen, namely, terms depending upon $\sin \phi - \theta$ and $\cos \phi - \theta$, and in all other cases whatever where constant terms can be produced. Where this is the case, therefore, the quantities in the equations for ω_1, &c., on the right side will vanish; but this will not be the case for such bodies as the earth, moon, &c., whose form, though nearly spherical, &c., differs from it on account of the irregularities of surface, &c.

Integration of the Differential Equations.

For this purpose the following equations for determining ϕ, &c., which are given in all dynamical treatises of motion about a fixed point, will be useful, viz.:—

$$\frac{d\phi}{dt} = \omega_3 - \frac{\cos\iota}{\sin\iota}\,(\omega_1 \sin\phi + \omega_2 \cos\phi)$$

$$\frac{d\psi}{dt} = -\frac{1}{\sin\iota}\,(\omega_1 \sin\phi + \omega_2 \cos\phi)$$

$$\frac{di}{dt} = \omega_1 \cos\phi - \omega_2 \sin\phi,$$

in which ψ is the longitude (measured backwards) of the *moveable* axis of x, referred to a fixed line.

The first of them will give us an approximate value for ϕ: to use it we must first find ω_3. Now, neglecting the disturbing force, as also products of $\omega_1\omega_2$, which are supposed very small, and are, moreover, multiplied by $B - A$, the third of equations (A) becomes

$$\frac{d\omega_3}{dt} = 0, \text{ or } \omega_3 = n$$

Neglecting, therefore, small quantities in the first of the equations of this number, it becomes

$$\frac{d\phi}{dt} = n, \quad \text{or } \phi = nt,$$

if the time is supposed to commence when the *R.A* of the example x is 0; also for θ it will suffice to put $n_1 + \epsilon$. Thus the equations for ω_1 and ω_2 become, on putting n for ω_3, nt for ϕ, &c.,

$$\frac{d\omega_2}{dt} + \frac{C-B}{A} n\omega_2 = \frac{1}{A} M_y \sin(\overline{n-n_1}\, t + \epsilon) - \frac{1}{A} N \cos(\overline{n-n_1}\, t + \epsilon)$$

$$\frac{d\omega_2}{dt} + \frac{A-C}{B} n\omega_2 = -\frac{1}{B} N \sin(\overline{n-n_1}\, t + \epsilon) + \frac{1}{B} M_x \cos(\overline{n-n_1}\, t + \epsilon)$$

If we differentiate the first of these, and for the value of $\frac{d\omega_2}{dt}$, which will occur in the result, substitute its value as derived from the second, we shall obtain the following equation, from which ω_2 has been eliminated:

$$\frac{d^2\omega_1}{dt^2} + \frac{C-B}{A}\,\frac{C-A}{B}\, n^2\omega_1 = \left(\frac{M_y}{A}\,\overline{n-n_1} - \frac{C-B}{AB} M_x n\right) \cos(\overline{n-n_1}\, t + \epsilon)$$
$$+ N\left(\frac{n-n^1}{A} + \frac{C-Bn}{AB}\right) \sin(\overline{n-n_1}\, t + \epsilon)$$

The terms introduced into ω_1 by the integration of this, will be

$$\frac{\frac{M_y}{A}\,\overline{n-n_1} - \frac{C-B}{AB} M_x n}{\frac{C-B}{A}\,\frac{C-A}{B}\, n^2 - (n-n_1)^2} \cos(\overline{n-n^1\, t} + \epsilon)$$

$$+ \frac{N\left(\frac{n-n_1}{A} + \frac{C-B}{AB} n\right)}{\frac{C-B}{A}\,\frac{C-A}{B}\, n^2 - (n-n^1)^2} \sin(\overline{n-n_1}\, t + \epsilon)$$

call this, for shortness,

$$A_1 \cos(\overline{n-n_1}\, t + \epsilon) + B_1 \sin(\overline{n-n_1}\, t + \epsilon)$$

In like manner the integration of the equation ω_2 will give

$$\omega_2 = A_2 \cos(\overline{n-n_1}\, t + \epsilon) + B_2 \sin(\overline{n-n_1}\, t + \epsilon)$$

where

$$A_2 = -\frac{N\left(\frac{n-n_1}{B} - \frac{A-C}{AB} n\right)}{\frac{C-B}{A}\,\frac{C-A}{B}\, n^2 - \overline{n-n_1}^2} \qquad B_2 = -\frac{\left(\frac{M_x}{B}\,\overline{xn_1} + \frac{A-C}{AB} M_y n\right)}{\frac{C-B}{A}\,\frac{C-A}{B}\, n^2 - \overline{n-n_1}^2}$$

The complete integral will also contain arbitrary quantities of the form

$$\epsilon \cos\left(\sqrt{\frac{C-B}{A}\,\frac{C-A}{B}}\, nt + \alpha\right).$$

For the earth, however, the constant e is quite insensible ; but for the moon, Laplace says that it is variable. However, it will not affect the recent inquiry in either case, and therefore may be dismissed for the present.

Formation of the Constant Quantities in the Differential Equations.

If, now, we multiply together the two values of ω_1 and ω_2, given above, we shall have

$$\omega_1\omega_2 = \{A_1 \cos(\overline{n - n_1}\, t + \epsilon) + B_1 \sin(\overline{n - n_1}\, t + \epsilon)\}$$

$$\{A_2 \cos(\overline{n - n_1}\, t + \varepsilon) + B_2 \sin(\overline{n - n_1}\, t + \epsilon)\}$$

$$= \tfrac{1}{2}A_1A_2 + B_1B_2) +$$

periodic terms; retaining the constant part

$$\frac{B - A}{C}\,\omega_1\omega_2 = \frac{1}{2}\,\frac{B - A}{C}\left(A_1A_2 + B_1B_2\right),$$

or, putting for A_1, &c., their values, and dividing numerator and denominator by

$$\overline{n - n^1}^2 - \frac{C - A\ C - B}{AB}\,n^2,$$

$$= \frac{1}{2}\,\frac{B - A}{ABC}\,\frac{N.(M_y + M_x)}{D.}$$

where

$$D = \frac{C - A\ C - B}{AB}\,n^2 - \overline{n - n_1}^2$$

This part, therefore, of the differential equation for ω_3 contains a constant term ; but before we can say that the entire equation does so, it is necessary to develop the term on the right side of the equation. Now, if this is expanded, it will be easily seen to consist entirely of sines and cosines, of which the general form may be said to be

$$I \sin(\overline{pn - qn_1}\, t + \varpi)$$

where I is some function of i, and p and q are whole numbers. It would appear, therefore, at first sight, to contain no constant term; but in reality it will be seen that it does. For, it is easily seen that every term in ω_1 and ω_2, such as those found above, will introduce a periodic term into the value of ι, as also of ϕ and θ, and the multiplication together of periodic terms may produce a constant. To see what terms in the development spoken of will be necessary, we must find the variations of i, ϕ and θ, then,

$$\frac{di}{dt} = \omega_1 \cos \phi - \omega_2 \sin \phi.$$

Put for ϕ its first approximate value nt

$$\omega_1 \cos \phi = \tfrac{1}{2} (A_1 \cos (\overline{2n - n_1}\, t + \epsilon) + B_1 \sin (\overline{2n - n_1}\, t + \epsilon) \\ + A_1 \cos (n_1 t + \epsilon) - B_1 \sin (nt + \epsilon)\}$$

$$\omega_2 \sin \phi = \tfrac{1}{2} (A_2 \sin (\overline{2n - n_1}\, t + \epsilon) - B_2 \cos (\overline{2n - n_1}\, t + \epsilon) \\ + A_2 \sin (n_1 t + \epsilon_1) + B_2 \cos (n_1 t + \epsilon)\}$$

taking the difference, and integrating,

$$\iota = i_1 + \tfrac{1}{2} \left(\frac{A_1 + B_2}{2n - n_1} \sin (\overline{2n - n_1}\, t + \epsilon) + \frac{A_2 - B_1}{2n - n_1} \cos (\overline{2n - n_1}\, t + \epsilon) \right) \\ + \tfrac{1}{2} \left(\frac{A_1 - B_2}{n_1} \sin (n_1 t + \epsilon) + \frac{A_2 + B_1}{n_1} \cos (n_1 t + \epsilon) \right\}$$

for the variations of ϕ we have

$$\frac{d\phi}{dt} = \omega_3 - \frac{\cos \iota}{\sin \iota} (\omega_1 \sin \phi + \omega_2 \cos \phi),$$

its variations, therefore, will arise partly from those of ω_3, and partly from those of ω_1 and ω_2. We may set aside the former for the present, and confine ourselves to those of ω_1 and ω_2.

Then, priming the function

$$\frac{\cos \iota}{\sin \iota} (\omega_1 \sin \phi + \omega_2 \cos \phi)$$

exactly in the same way as above, and integrating, we have

$$\phi = nt + \frac{1}{2} \frac{\cos \iota}{\sin \iota} \left\{ \frac{A_1 + B_2}{2n - n_1} \cos (\overline{2n - n_1} t + \varepsilon - \frac{A_2 - B_1}{2n - n_1} \sin (\overline{2n - n_1}\, t + \varepsilon) \right\} \\ + \frac{1}{2} \frac{\cos \iota}{\sin \iota} \left\{ \frac{A_1 - B_2}{n_1} \cos (n_1 t + \epsilon) - \frac{A^2 + B_1}{n_1'} \sin (n_1 t + \epsilon) \right\}$$

also since the differential equation for ψ is

$$\frac{d\psi}{dt} = - \frac{1}{\sin \iota} (\omega_1 \sin \phi + \omega_2 \cos \phi)$$

the variations of ψ will be the same as those of ϕ, only not multipled by $\cos \iota$.

We must, therefore, carefully seek out all terms in the development of it, whose argument is either $2\phi - \theta$ or θ. The latter, however, will be found to disappear. Then,

$$N_1 \frac{\mu}{r^1} \Sigma(yx_1 - xy_1) \left\{1 - \&c. + \frac{15}{2}\left(\frac{xx_1 + yy_1 + z_1 z_1}{r^2}\right)^2\right\} N_1 = \&c.$$

$$= \cdot \frac{\mu}{\&c.} \Sigma(yx_1 - xy_1) \frac{15}{2}\left(x^2x_1^2 + y^2y_1^2 + z^2z_1^2 + 2xy\, x_1y_1 + 2xz\, x_1z_1 + 2yz\, y_1z_1\right)$$

$$= \frac{1}{C} \cdot 15 \frac{\mu}{r.^4} \left\{\cdot \overline{x_1y_1z_1}\,(y^2 - x^2)z + (\overline{x_1^2 - y_1^2})\, xyz\right\}$$

$$= + \frac{1}{C} \frac{\mu}{r^4} 15 \left(a^2 \cos \overline{2\phi - 2\theta} + 2a\beta \cos 2\phi + \beta^2 \cos \overline{2\phi + 2\theta}\right)$$

$$\overline{x\, y_1z_1} \sin \iota \sin \theta$$

$$+ \frac{1}{C} \frac{\mu}{r^4} \frac{15}{2}\left(a^2 \sin \overline{2\phi - 2\theta} + a\beta \sin 2\phi + \beta^2 \sin \overline{2\phi + 2\theta}\right)(\overline{x_1^2} - \overline{y_1^2})$$

$$\sin \iota \sin \theta$$

$$= \frac{1}{C} \frac{\mu}{r^4} \frac{15}{2} (a^2 - 2a\beta)\, \overline{x_1y_1z_1} \sin \iota \sin \overline{2\phi - \theta}$$

$$- \frac{1}{C} \frac{\mu}{r^4} \frac{15}{4} (a^2 - 2a\beta)\, (\overline{x_1^2} - \overline{y_1^2}) \sin \iota \cos \overline{2\phi - \theta})$$

Now, taking the value of i found above, we shall have

$$\sin \iota = \sin \left(\iota_1 + \tfrac{1}{2} \frac{A_1 + B_2}{2n - n^1} \sin \overline{2n - n_1} + -\right)$$

neglecting, for the present, the other term,

$$= \sin \iota_1 + \tfrac{1}{2} \cos \iota_1 \frac{A_1 + B_2}{2n - n^1} \sin \overline{2n - n_1} +$$

$$\cos \iota = \cos \iota_1 - \tfrac{1}{2} \sin \iota_1 \frac{A_1 + B_2}{2n - n} \sin \overline{2n - n_1} -, \&c.$$

Then, since a stands for $\frac{1 + \cos \iota}{2}$, we shall have

$$a^2 = \tfrac{1}{4}\left(\overline{1 + \cos \iota_1}^2 - \overline{1 + \cos \iota_1} \sin \iota_1 \frac{A_1 + B_2}{2n - n^1} \overline{A_1 + B_2} \sin \overline{2n - n^1}\right))$$

and

$$- 2a\beta = - \tfrac{1}{2} \sin^2 \iota = - \tfrac{1}{2}\left(\sin^2 \iota_1' + \sin \iota_1 \cos \iota_1 \frac{A_1 + B_2}{2n - n_1} \sin 2n - n^1\right)$$

$$\therefore\ a^2 - 2a\beta = \tfrac{1}{4}\left(\overline{1 + \cos\iota_1}^2 - 2\sin^2\iota_1\right) - \tfrac{1}{4}\sin\iota_1\,(1 + 3\cos\iota_1)$$

$$\frac{A_1 + B_2}{2n - n^1}\sin\overline{2n - n^1}.$$

Multiplying this by the value of sin *i*, we have

$$(a^2 - 2a\beta)\sin\iota = \tfrac{1}{4}\left(\overline{1 + \cos\iota_1}^2 - 2\sin^2\iota_1\right)\sin\iota_1$$

$$+\left(\tfrac{1}{8}\left(\overline{1 + \cos\iota_1}^2 - 2\sin^2\iota_1\right)\cos\iota_1 - \tfrac{1}{4}\sin^2\iota_1\,(1 + 3\cos\iota_1)\right)$$

$$\frac{\overline{A_1 + B_2}}{2n - n^1}\sin\overline{2n - n_1}.$$

Also, considering the variations introduced into ϕ, we have

$$\phi = nt + \tfrac{1}{2}\frac{\cos\iota_1}{\sin\iota_1}\,\frac{A_1 + B_2}{2n - n^1}\cos\overline{2n - n^1}$$

also θ is the true longitude measured from the moveable axis whose longitude measured backwards is ψ; therefore if $n_1 t$ be the longitude measured from a fixed axis, that measured from the moveable axis will be $n^1 t + \psi$; or, from the value of ψ given above,

$$n^1 t + \frac{1}{2}\,\frac{1}{\sin\iota_1}\,\frac{A_1 + B_2}{2n - n^1}\cos\overline{2n - n^1},$$

$$\therefore\ 2\phi - \theta = 2nt - n^1 t + \tfrac{1}{2}\,\frac{2\cos\iota_1 - 1}{\sin\iota_1}\,\frac{A_1 + B_2}{2n - n^1}\cos\overline{2n - n^1},$$

and

$$\sin\overline{2\phi - \theta} = \sin\overline{2n - n^1} + \tfrac{1}{4}\,\frac{2\cos\iota_1 - 1}{\sin\iota_1}\,\frac{\overline{A_1 + B_2}}{2n - n^1}.$$

Multiplying this by the value of $(\alpha^2 - 2a\beta)\sin\iota$, we shall have for the constant term depending upon $\overline{A_1 + B_2}$,

$$\left\{\frac{1}{16}\left(\overline{1 + \cos\iota_1}^2 - 2\sin^2\iota_1\right)(3\cos\iota_1 - 1) - \frac{1}{8}\sin^2\iota_1\,(1 + 3\cos\iota_1)\right\}\frac{A_1 + B_2}{2n - n^1}$$

which may be put into the simpler form

$$\frac{1}{4}\left(1 + \cos\iota_1 - \frac{5}{4}\sin^2\iota_1\,(1 + 3\cos\iota_1)\right)\frac{A_1 + B_2}{2n - n^1}$$

and in like manner the quantity $(a^2 + 2a\beta)\sin\iota$ will contain the constant quantity

$$\frac{1}{4}\left(1 + \cos\iota_1 - \frac{5}{4}\sin^2\iota_1\,(1 + 3\cos\iota_1)\right)\frac{A_2 - B_1}{2n - n^1}$$

On substituting these values, the right-hand side of the equation for determining ω_3 contains the constant terms

$$\frac{1}{C}\frac{\mu}{r^4}\frac{15}{8}\left(1+\cos_1-\frac{5}{4}\sin^2\iota\,(1+3\cos\iota)\right)\left\{\overline{x_1y_1z_1}\,\frac{A_1+B_2}{2n-n^1}\right.$$

$$\left.-\tfrac{1}{2}(\overline{x_1^2}-\overline{y_1^2})\,\frac{A_2-B_1}{2n-n^1}\right\}$$

This latter factor may be more conveniently put into the form

$$\overline{x_1y_1z_1}\,\frac{A_1}{2n-n_1}+\tfrac{1}{2}\,\overline{x_1^2}-\overline{y_1^2}\,\frac{B_1}{2n-n^1}+\overline{x_1y_1z_1}\,\frac{B_2}{2n-n^1}-\tfrac{1}{2}\,\overline{x_1^2}-\overline{y_1^2}\,\frac{A_2}{2n-n^1}$$

Let us first find the value of the first pair of these terms. Now,

$$A_1=\frac{1}{D}\left(\frac{M_y}{A}\,\overline{n-n^1}-\frac{C-B}{AB}M_x\,n\right)$$

and

$$B_1=\frac{1}{D}N\left(\frac{n-n^1}{A}+\frac{C-B}{A}n\right)$$

where

$$M_y=\frac{\mu}{r^4}\left\{\frac{15}{8}\left((a^3+2\beta^2a)\,\overline{x_1^2z_1}+3\,(a^3+2\beta^2a)-\sin^2\iota\,\overline{4a-2\beta}\right)\overline{y_1^2z_1}\right.$$

$$+\sin^2\iota\,\overline{2a-\beta}\,\overline{z_1^3}-\frac{3}{Z}\,\overline{ar_1^2z_1}$$

$$N=\frac{15}{4}\,\frac{\mu}{r^4}\,(a^3+2a\beta^2-\overline{2a-\beta}\sin^2\iota)\,\overline{x_1y_1z_1}$$

then

$$A_1\,\overline{x_1y_1z_1}+\tfrac{1}{2}\,(\overline{x_1^2-y_1^2})\,B_1$$

becomes, considering first the term multiplied by $\overline{n-n^1}$,

$$\left\{\frac{\mu}{r^4}\left\{\frac{15}{8}\left((a^3+2\beta^2a)\,\overline{x_1^2z_1}+(3(a^3+2\beta^2a)-\sin^2\iota\,\overline{4a-2\beta})\,\overline{y_1^2z_1}\right.\right.\right.$$

$$\left.+\sin^2\iota\,\overline{2a-\beta z_1}^{3}\right)-\frac{1}{2}\,\overline{ar_1^2z_1}\,\overline{x_1y_1z_1}+\frac{\mu}{r^4}\left\{\frac{15}{8}\,(a^3+2\beta^2a\right.$$

$$\left.-\overline{2a-\beta}\,\sin^2\iota)\,(\overline{x_1^2z^1}-\overline{y_1^2z_1})\right\}\frac{n-n^1}{AD}$$

or

$$\frac{\mu}{r^4}\left(\frac{15}{8}\left\{(2(a^3+2\beta^2a)-\overline{2a-\beta}\,\sin^2\iota)\,(\overline{x_1^2z_1}+\overline{y_1^2z_1})+\sin^2\iota\,\overline{2a-\beta z_1}^{3}\right\}\right.$$

$$\left.-\frac{3}{4}\,\overline{ar_1^2z_1}\right)x_1y_1z_1\,\frac{n-n^1}{AD}$$

This coefficient, it is easily seen, is equal to $\frac{1}{2}(M_x + M_y)$, so that we may replace the above expression by

$$\tfrac{1}{2}\,\overline{M_x + M_y}\;\overline{x_1y_1z_1}\,\frac{n - n^1}{A}$$

in like manner it will be seen that the part multiplied by n becomes

$$\frac{1}{2}\,\frac{M_x + M_y}{D}\;\overline{x_1y_1z_1}\;\frac{B - C}{AB}\,n$$

thus, the two terms in question become

$$\frac{1}{2}\,\frac{M_x + M_{y1}}{D}\;\frac{x_1y_1z_1}{2n - n_1}\left(\frac{n - n_1}{A} + \frac{B - C}{AB}\,n\right)$$

and in like manner the latter pair become

$$-\frac{1}{2}\left(\frac{M_x + M_y}{D}\right)\frac{\overline{x_1y_1z_1}}{2n - n_1}\left(\frac{n - n^1}{B} + \frac{A - C}{AB}\,n\right)$$

therefore the sum of the four is

$$\frac{1}{2}\,\frac{M_x + M_y}{D}\;\frac{x_1y_1z_1}{2n - n^2}\left(\overline{n - n}\left(\frac{1}{A} - \frac{1}{B}\right) + \frac{B - B}{AB}\,n\right)$$

$$= \frac{1}{2}\,\frac{\overline{B - A}}{AB}\;\overline{M_x + M_y}\;\overline{x_1y_1z_1}$$

thus the constant terms of the right-hand side of the equation for ω_3 become

$$\frac{\mu}{r^4}\,\frac{15}{8}\left(1 + \cos\iota - \frac{5}{4}\sin^2\iota\;\overline{1 + 3\cos\iota}\right)\overline{x_1y_1z_1}\;\frac{1}{2}\,\frac{\overline{B - A}}{ABC}\left(\frac{M_x + M_{y1}}{D}\right)$$

also on putting for α and β their values, it is easily seen that the value of N is

$$\frac{15}{8}\,\frac{\mu}{r^4}\left(1 + \cos\iota - \frac{5}{4}\sin^2\iota\;\overline{1 + 3\cos\iota}\right)\overline{x_1y_1z_1}$$

so that the expression just found for the constant term in the development of N reduces itself to

$$\frac{1}{2}\,\frac{\overline{B - A}}{ABC}\left(\frac{M_x + M_y}{D}\right)N.$$

And this is identical with that part on the other side of the equation for $\frac{d\omega_3}{dt}$, which arises from the multiplication together of ω_1 and ω_2, so that these terms identically destroy each other.

No account, however, has as yet been taken of the first terms which occur in the development of L, namely,

$$-\frac{3}{2}\frac{\mu}{r^3}\frac{B-A}{C}\left\{a^2 \sin \overline{2\phi-2\theta} + 2a\beta \sin 2\phi + \beta^2 \sin 2\phi + 2\theta\right\}$$

Now, it is evident that these will contain constant terms, which, if they do not identically destroy each other, or are destroyed by terms which may arise in other parts of the differential equations, will give terms indicating a gradual and permanent change of motion. Let us see how constant terms might arise in the above expression.

In the first place it has been already seen that the expression for i contains, amongst others, terms of the form

$$H \sin \overline{2n-n^1}, \quad K \cos n^1, \text{ \&c.,}$$

we shall thus have

$$\cos \iota = \cos (i_1 + H \sin \overline{2n-n^1} + K \cos n_1) = \cos (\iota_1 + P)$$

suppose

$$= \cos \iota_1 \left\{1 - \tfrac{1}{2} P^2 + \gamma\right\} + \sin \iota_1 \left(P - \frac{P^3}{3} - \gamma\right)$$

$$= \cos \iota_1 \left\{1 - H \sin \overline{2n-n^1}\ K \cos n^1\right\} + P \sin \iota_1, \text{ \&c.}$$

$$= \cos \iota_1 \left\{1 - \tfrac{1}{2} HK \sin \overline{2n-2n^1} + \tfrac{1}{2} HK \sin 2n\right\} + P \sin \iota_1.$$

In like manner if $\sin (2\phi - 2\theta)$ be developed, it will contain the term $\sin \overline{2n-2n^1}$, terms multiplied by P, and constant terms. The first of which, when multiplied by $HK \sin \overline{2n-2n^1}$, will produce a constant term, and the second when multiplied by $P \sin i_1$, and the third when multiplied by $\cos \iota_1$.

Again, the first approximate values already found for $i_1 \phi$ and θ, will, by substitution in the differential equations, produce terms having the arguments

$$2n - 2n_1, \quad 2n, \text{ \&c., in } \iota\ \phi \text{ and } \psi.$$

And these will arise in two ways: first, it is evident that such a term as $H \cos n^1$ occurring in the expression for i, would, when introduced into a or $\cos i$, where it occurs in the terms in the differential equations already used for determining terms in ω_1, &c., having the argument $\phi - \theta$ or $\overline{n - n^1}$, would introduce terms into ω_1, &c., having the argument $n - 2n_1$, and these, when multiplied by $\cos \phi$, where it occurs in the equation for i, viz. $\omega_1 \cos \phi$, would produce terms having $2n - 2n^1$ for their argument in the expression for i. These terms, however, will be multiplied by higher powers of $\sin i$ than those which arise in the manner about to be examined, and therefore for a first approximation, at least, may be neglected, especially in cases where i is small; and even where it is not, it will be best to reserve them, and, if necessary, to take them into account after lower powers have been examined.

The other, and more important way in which such terms arise is as follows:—Take the value already found for ϕ_1, viz.—

$$\begin{aligned} nt + \frac{1}{2}\,\frac{\cos\iota_1}{1\sin\iota_1}\,\Bigg\{ & \frac{A_1-B_2}{2n-n^1}\cos\overline{2n-n^1} + \frac{A_1-B_2}{n^1}\cos n^1 \\ & - \frac{A_2-B_1}{2n-n^1}\sin\overline{2n-n^1} - \frac{A_2+B_1}{n^1}{}^1\sin n^1\Bigg\} \end{aligned}$$

this will give

$$\begin{aligned} \cos\phi = \cos n - \frac{1}{4}\,\frac{\cos\iota_1}{1\sin\iota_1}\,\Bigg\{ & \left(\frac{A_1-B_2}{n^1} - \frac{A_1+B_2}{2n-n^1}\right) \\ & \sin\overline{n-n^1} - \left(\frac{A_2-B_1}{2n-n^1} + \frac{A_2+B_1}{n^1}\right)\cos\overline{n-n^1} \end{aligned}$$

Multiply this by $\omega_1 = A_1\cos\overline{n-n^1} + B_1\sin\overline{n-n^1}$, and retain the terms involving $\cos\overline{2n-2n^1}$, and we have

$$\begin{aligned} \omega_1\cos\phi = \frac{1}{8}\,\frac{\cos\iota_1}{1\sin\iota_1}\,\Bigg\{ & \left(\frac{A_1-B_2}{n^1} - \frac{A_1+B_2}{2n-n^1}\right)B_1 \\ & + \left(\frac{A_2-B_1}{2n-n^1} + \frac{A_2+B_1}{n^1}\right)A_1\Bigg\}\cos\overline{2n-2n^1} \end{aligned}$$

in like manner we shall find

$$\begin{aligned} \omega_2\sin\phi = \frac{1}{8}\,\frac{\cos\iota_1}{\sin\iota_1}\,\Bigg\{ & \left(\frac{A_1+B_2}{2n-n^1} + \frac{A_1-B_2}{n^1}\right)A_2 \\ & + \left(\frac{A_2-B_1}{2n-n^1} - \frac{A_2+B}{n^1}\right)B_2\Bigg\}\cos\overline{2n-2n^1} \end{aligned}$$

$$\begin{aligned} \therefore \frac{d\iota}{dt} = \omega_1\cos\phi - \omega_2\sin\phi = \frac{1}{8}\,\frac{\cos\iota}{\sin\iota}\,\Bigg\{ & \frac{\overline{A_1-B_2}\;\overline{A_2-B_1}}{2n-n^1} + \frac{\overline{A_1+B_2}\;\overline{A_2+B_1}}{n^1} \\ & - \frac{\overline{A_2-B_1}\;\overline{A_1-B_2}}{n^1} - \frac{\overline{A_2+B_1}\;\overline{A_1+B_2}}{2n-n^1}\Bigg\}\cos\overline{2n-2n^1} \end{aligned}$$

$$= \frac{1}{8}\,\frac{\cos\iota}{\sin\iota}\left(\overline{A_2+B_1}\;\overline{A_1+B_2} - \overline{A_1-B_2}\;\overline{A_2-B_1}\right)\frac{2n-2n^1}{(2n-n^1)\,n^1}\cos\overline{2n-2n_1}$$

$$\begin{aligned} \therefore \iota - \iota_1 + - + \frac{1}{8}\,\frac{\cos\iota}{\sin\iota}\,\Big\{ & \overline{A_2+B_1}\;\overline{A_1+B_2} - \overline{A_1-B_2}\;\overline{A_2-B_1}\Big\} \\ & \frac{1}{(2n-n^1)\,n^1}\sin(2n-2n^1) \end{aligned}$$

To find $\overline{\phi - \theta}$ we have

$$-\frac{d(\phi - \psi)}{dt} = \frac{\cos\iota - 1}{\sin\iota}(\omega_1 \sin\phi + \omega_2 \cos\phi)$$

and by proceeding exactly as above we shall find

$$\omega_1 \cos\phi + \omega_2 \sin\phi = \tfrac{1}{2}(A_1 + B_2)\sin\overline{2n - n^1} + \tfrac{1}{2}\overline{A_1 - B_2}\sin n^1$$

$$+ \tfrac{1}{2}\overline{A_2 - B_1}\cos\overline{2n - n^1} + \tfrac{1}{2}\overline{A_2 + B_1}\cos n^1$$

$$+ \frac{1}{8}\frac{\cos\iota_1}{\sin\iota_1}(\overline{A_1 + B_2}\,\overline{A_2 + B_1} - \overline{A_2 - B_1}\,\overline{A_1 - B_2})\left(\frac{1}{2n - n^1} + \frac{1}{n^1}\right)$$

$$\sin(2n - 2n^1)$$

also if $\iota = \iota_1 + \Sigma$ where Σ denotes the sum of the quantities which occur in it, we shall have

$$\frac{\cos\iota - 1}{\sin\iota} = \frac{\cos\iota - \sin\iota . \Sigma - 1}{\sin\iota_1 + \cos\iota\,\Sigma} = \frac{\cos\iota_1 - 1 - \sin\iota_1\,\Sigma}{\sin\iota_1\left(1 + \frac{\cos\iota_1}{\sin\iota_1}\Sigma\right)}$$

$$= \frac{\cos\iota_1 - 1}{\sin\iota_1} - \Sigma . \left(1 - + \frac{\overline{\cos\iota_1 - 1}\cos\iota_1}{\sin^2\iota_1}\right),$$

neglecting higher powers of Σ than the first, since these only are wanted. Multiplying together these factors, we shall have

$$-\frac{d(\phi - \psi)}{dt} = \frac{\cos\iota_1 - 1}{\sin\iota_1}\left(\tfrac{1}{2}\overline{A_1 + B_2}\sin\overline{2n - n^1} +, \&c.\right)$$

$$-(\overline{A_1 + B_2}\,\overline{A_2 - B_1} - \overline{A_2 - B_1}\,\overline{A_1 - B_2})\left(1 + \frac{\overline{\cos\iota_1 - 1}\cos\iota_1}{\sin^2\iota_1}\right.$$

$$\left. - \frac{\overline{\cos\iota_1 - 1}\cos\iota_1}{\sin^2\iota_1}\right)\frac{2n}{2n - n^1}\sin\overline{2n - 2n^1}$$

Integrating and changing the signs on both sides

$$\phi - \theta = \frac{\cos\iota - 1}{\sin\iota_1}\,\frac{1}{2}\left(\frac{A_1 + B_2}{2n - n^1}\cos\overline{2n - n^1} + ..\right)$$

$$-\tfrac{1}{8}(\overline{A_1 + B_2}\,\overline{A_2 + B_1} - \overline{A_1 - B_2}\,\overline{A_2 - B_1})\frac{n}{\overline{n - n^1}\,\overline{2n - n^1}\,n^1}\cos(2n - 2n^1)$$

We have now to substitute these values for ι_1, &c., in the function

$$-\frac{3}{8}\frac{\mu}{r^3}(1 + \cos\iota)^2 \sin\overline{2\phi - 2\theta}$$

which occurs in the development of N.

In the first place,

$$\cos\iota = \cos\left(\iota_1 + \Sigma + \frac{1}{8}\frac{\cos\iota}{\sin\iota}\left(\frac{\overline{A_1+B_2}\;\overline{A_2+B_1} - \overline{A_1-B_2}\;\overline{A_2-B_I}}{(2n-n^1)\,n^1}\right.\right.$$

$$\left.\left.\sin\overline{2n-6n^1}\right)\right)$$

$$= \cos\iota_1\left\{1-\tfrac{1}{2}\Sigma^2\right\} - \sin\iota_1\,\Sigma - \tfrac{1}{8}\cos\iota\,\frac{\overline{A_1+B_2}\;\overline{A_2+B_1} - \overline{A_1-B_2}\;\overline{A_2-B_1}}{(2n-n^1)\,n^1}$$

$$\sin\overline{2n-6n^1}$$

Now,

$$\Sigma^2 = \left(\frac{1}{2}\,\frac{\overline{A_1+B_2}}{2n-{}^1}\sin\overline{2n-n^1} + \frac{1}{2}\,\frac{A_2+B_1}{n^1}\cos n_1\,\tau\,1\right)^2$$

$$= \frac{1}{4}\,\frac{\overline{A_1+B_2}\;\overline{A_2+B_1} - \overline{A_1-B_2}\;\overline{A_2-B_1}}{(2n-n^1)\,n^1}\sin\overline{2n-n^1} + \tau,$$

$$\therefore \cos\iota = \cos\iota_1 - \sin\iota_1\,\Sigma - \tfrac{1}{4}\cos\iota_1\,\frac{P}{(2n-n^1)\,n^1}\sin\overline{2n-2n^1}$$

where $P = \overline{A_1+B_2}$, &c.

$$\overline{1+\cos\iota}^2 = (1+\cos\iota_1)^2 - 2\,\overline{1+\cos\iota_1}\,\sin\iota_1\,\Sigma - \tfrac{1}{2}\,\overline{1+\cos\iota_1}\,\cos\iota_1$$

$$\frac{P}{(2n-n^1)\,n^1}\sin\overline{2n-2n^1} + \sin\iota\,\Sigma^2$$

$$= \overline{1+\cos\iota_1}^2 - 2\,\overline{1+\cos\iota}\,\sin\iota_1\,\Sigma + \left(\tfrac{1}{4}\sin^2\iota - \tfrac{1}{4}\,\overline{1+\cos\iota_1}\,\cos\iota_1\right)\frac{P}{(2n-n^1)\,n^1}$$

$$\sin(2n-n^1)\}$$

also

$$\sin\overline{2\phi-2\theta} = \sin\overline{2n-2n^1}\left\{1-\tfrac{1}{2}\,\frac{\overline{\cos\iota_1-1}^2}{\sin\iota_1}\,(2\Sigma_1)^2\right\}$$

$$+\,2\,\frac{\cos\iota_1-1}{\sin\iota_1}\cos\overline{2n-2n^1},\;\; \Sigma_1 - \tfrac{1}{4}P\,\frac{n}{\overline{2n-n^1}\;\overline{n-n^1}\;n^1}\cos^2(2n-2n^1)$$

where Σ_1 is the sum of the terms found in the first approximation. On expanding $\overline{2\Sigma^2}$, this becomes

$$\sin\overline{2n-2n^1}\left\{1-\tfrac{1}{2}\,\frac{\overline{\cos\iota_1-1}^2}{\sin\iota}\,\frac{P}{\overline{2n-n^1}\;n^1}\sin\overline{2n-2n^1}\right\} +,\ \&c.$$

$$= \sin\overline{2n-2n^1} + \frac{2\cos\iota-1}{\sin\iota}\cos\overline{2n-2n^1}\,.\,\Sigma_1 - \left(\frac{1}{4}\,\frac{\overline{\cos\iota-1}^2}{\sin^2\iota}\,\frac{1}{(2n-n^1)\,n^1}\right.$$

$$\left. + \frac{1}{8}\,\frac{n}{\overline{2n-n}\;\overline{n-n^1}\;n^1}\right)P.$$

If now we put for Σ. and Σ_1 their values, and multiply this by the value found above for $\overline{1 + \cos . \iota^2}$, we shall have, observing that the constant terms in $\Sigma\Sigma_1 \cos \overline{2n - 2n^1}$ destroy each other, and retaining the constant term,

$$(1 + \cos \iota)^2 \sin \overline{2\phi - 2\theta} = \left\{ \left(- \frac{1}{4} \frac{\overline{\cos \iota_1 - 1}^2 \, \overline{\cos \iota_1 + 1}^2}{\sin^2 \iota} + \frac{1}{8} \right. \right.$$

$$\left. \left. \sin^2 \iota_1 - \frac{1}{4} \overline{1 + \cos \iota_1} \cos \iota_1 \right) \frac{1}{\overline{2n - n^1}\, n^1} - \frac{1}{8} \overline{1 + \cos \iota_1}^2 \frac{n}{(2n - n)(n - n^1)\, n^1} \right\} P$$

the former part may be put into the form

$$\overline{\tfrac{1}{8} - \tfrac{1}{4}} \sin^2 \iota - \tfrac{1}{8} \overline{1 + \cos \iota}\; 2 \cos \iota_1$$

or

$$- \tfrac{1}{8} \sin^2 \iota - \tfrac{1}{8} \overline{4 \cos \iota} \,(1 + \cos \iota_1 - \overline{1 - \cos \iota_1}) \quad \text{or} \quad - \tfrac{1}{8} \overline{1 + \cos \iota}^2$$

therefore the whole expression becomes

$$- \tfrac{1}{8} \overline{1 + \cos \iota_1}^2 \left(\frac{n}{(2n - n^1)(n - n^1)\, n^1} + \frac{1}{\overline{2n - n^1}\, n^1} \right) P$$

or

$$- \tfrac{1}{8} \overline{1 + \cos \iota_1}^2 \frac{1}{(n - n^1)\, n^1} P.$$

Therefore, finally, the function which we have been examining contains the constant term

$$\left(\frac{B - A}{C} \right) \frac{\mu}{r^3} \frac{3}{8} \frac{1}{8} \overline{1 + \cos \iota}^2 \frac{1}{(n - n^1)\, n^1} (\overline{A_1 + B_2}\; \overline{A_2 + B_1} - \overline{A_1 - B_2}\; \overline{A_2 - B_1})$$

or, as it may be more shortly written

$$\left(\frac{B - A}{C} \right) \frac{3}{8} \frac{\mu}{r^3} \frac{\overline{1 + \cos \iota}^2}{4} \frac{1}{(n - n^1)\, n^1} (A_1 B_1 + A_2 B_2).$$

On putting for A_1, &c., their values, and multiplying out, the latter factor becomes

$$\frac{N}{D^2} \left(\frac{M_y}{A^2} + \frac{M_y}{B^2} \right) \overline{(n - n^1}^2 - \frac{N}{D^2} \left(\frac{M_y \overline{A - C}^2 + M_x \overline{C - B}^2}{A^2 B^2} \right) n^2$$

$$+ \frac{N}{D^2} \left(M_y \left(\frac{C - B}{A^2 B} + \frac{A - C}{B^2 A} \right) - M_x \left(\frac{A - C}{C B^2} + \frac{C - B}{A^2 B} \right) \right) n \, \overline{n - n_1}$$

This may be much simplified for bodies nearly spherical, and might be put into the form

$$\frac{N}{D^2} \frac{M_y + M_x}{AB} \left(\overline{n - n_1}^2 - \frac{\overline{C - A}\; \overline{C - B}}{AB} n^2 \right)$$

if we neglect small quantities which will be multiplied by $B - A$, &c.), or observing that the latter factor is the same as D, it becomes simply

$$\frac{N}{D}\frac{M_x + M_y}{AB},$$

and the term becomes

$$\frac{B-A}{ABC}\,\frac{3}{8}\,\frac{\overline{1+\cos\iota_1}^2}{4}\,\frac{\mu}{r^3}\,\frac{1}{(n-n^1)\,n^1}\,\frac{N}{D}\left(M_x + M_y\right), \qquad (a)$$

which is true for every value of ι. Suppose, for simplicity, that $\cos\iota$ does not much differ from unity; then on this supposition we should have

$$N = \frac{15}{4}\,\frac{\mu}{r^4}\,\overline{x_1 y_1 z}. \quad M_y = \frac{\mu}{r^4}\left\{\frac{15}{8}\left(\overline{x_1^2 z_1} + 3\overline{y_1^2 z_1}\right) - \frac{3}{2}\overline{r_1^2 z_1}\right\} \text{ \&c.}$$

$$\therefore\ M_x + M_y - \frac{\mu}{r^4}\left(\frac{15}{8}\overline{x_1^2 z_1} + \overline{y_1^2 z_1} - 3r_1^2 z_1^1\right) = \frac{\mu}{r^4}\,\frac{3}{2}\left(3\left(\overline{x_1^2 z_1} + \overline{y_1^2 z_1}\right) - \overline{z_1^3}\right)$$

upon substituting these values for N_1 and $M_x + M_y$, the expression a becomes

$$\frac{B-A}{ABC}\,\frac{135}{64}\left(\frac{\mu}{r^3}\right)^3\frac{1}{(n-n^1)\,n^1}\,\frac{1}{D}\,\frac{\left(3\left(\overline{x_1^2 z_1} + \overline{y_1^2 z_1}\right) - \overline{z_1^3}\right)}{r^2}\,\overline{x_1 y_1 z_1} \qquad (A)$$

This term, it will be observed, is multiplied by the cube of $\frac{\mu}{r^3}$, whereas the terms which have been previously examined were only multiplied by the square. If now we resume the expression in the development of N which is multiplied by

$$\frac{\mu}{r^4}\left\{\overline{x_1 y_1 z_1}\,\frac{A_1 + B_2}{2n - n^1} - \tfrac{1}{2}\left(\overline{x_1^2} - \overline{y_1^2}\right)\frac{A_2 - B_1}{2n - n^1}\right\}$$

we shall see that, on continuing the approximation, $(A_1 + B_2)$ and $(A_2 - B_1)$ will receive an increment such that when their new value is substituted for them in the above expression, it will be multiplied by the cube of $\frac{\mu}{r^3}$. The term which so arises will not destroy that previously found, but will be of the same order, and will modify it; therefore it must be sought out. To find it,

Let us return to the original equations for ω_1 ω_2, namely,

$$\frac{d\omega_1}{dt} + \cdot = \frac{3}{2}\,\frac{\mu}{r^3}\,\frac{C-B}{A}\sin\iota\left\{\frac{1+\cos\iota}{2}\cos\overline{\psi - 2\theta} - \cos\iota\cos\phi\right.$$

$$\left. - \frac{1-\cos\iota}{2}\cos\overline{\phi + 2\theta}\right\}$$

$$\frac{d\omega_2}{dt} + . = \frac{3}{2}\,\frac{\mu}{r^3}\,\frac{A - C}{B}\sin\iota\left\{\frac{1+\cos\iota}{2}\sin\overline{\phi - 2\theta} - \ldots\right\}$$

If, now, we take the terms already found in the first approximation for ι, ϕ, $\nu\psi$, viz.,

$$\iota = \iota_1 + \tfrac{1}{2}\,\frac{A_1 - B_2}{n^1}\sin n^1 + \tfrac{1}{2}\,\frac{A_2 + B_1}{n^1}\cos n^1, \text{ \&c.}$$

and substitute them for ι and ϕ, &c., in the above equations, we shall have terms whose argument is $\overline{n - n^1}$. And it will suffice for the present if we substitute the values in the first term which occurs on the right-hand side of each of these equations; viz. that depending upon $\phi - 2\theta$; for, although that depending upon ϕ, that is the second, would produce terms of the same form, yet the increment so formed is best considered in conjunction with the term in N which corresponds to it, namely, that depending upon sin 2ϕ. Making, therefore, the substitutions mentioned, we shall have to find the increment of ω_1,

$$\frac{d\omega_1}{dt} + \ldots = \frac{3}{8}\,\frac{\mu}{r_4}\,\frac{C - B}{A}\left(1 + \cos\iota - \tfrac{1}{2}\sin^2\iota\right)\left\{\frac{A_1 - B_2}{n^1}\sin\overline{n - n^1} + \frac{A_2 + B_1}{n^1}\cos\overline{n - n^1}\right\}.$$

This might be integrated in the same way that the equations for ω_1, &c., were treated above; but as it will suffice for the present purpose to reject quantities multiplied by products of

$$\frac{C - B}{A} \text{ and } \frac{C - A}{B},$$

we may integrate it without introducing the second differential coefficients. It will give

$$\omega_1 = \frac{3}{8}\,\frac{\mu}{r^3}\,\frac{C - B}{A}\left(1 + \cos\iota - \tfrac{1}{2}\sin^2\iota\right)\left\{-\frac{A_1 - B_2}{(n - n^1)\,n^1}\cos\overline{n - n^1} + \frac{A_2 + B_2}{(n - n^1)\,n^1}\sin\overline{n - n^1}\right\}$$

which, if we make the same supposition as to i which was made before, becomes

$$\omega_1 = \frac{3}{4}\,\frac{\mu}{r^3}\,\frac{C - B}{A}\left\{-\frac{\overline{A_1 + B_2}}{\overline{n - n^1}\,n^1}\cos\overline{n - n^1} + \frac{A_2 + B_1}{(n - n^1)\,n^1}\sin\overline{n - n^1}\right\}$$

for the increment of ω_1 above spoken of.

Similarly, for the increment of ω_2 we have

$$\omega_2 = \frac{3}{4}\frac{\mu}{r^3}\frac{C-A}{B}\left\{\frac{A_2+B_1}{(n-n^1)\,n^1}\cos\overline{n-n^1}\;\frac{A_1-B_2}{(n-n^1)\,n^1}\sin\overline{n-n^1}\right\}$$

that is, A_1 is increased by

$$\frac{3}{4}\frac{\mu}{r^3}\frac{C-B}{A}\left\{-\frac{A_1-B_2}{(n-n^1)\,n^1}\right\}$$

and B_2 by

$$\frac{3}{4}\frac{\mu}{r^3}\frac{C-A}{B}\left(\frac{A_1-B_2}{(n-n^1)\,n^1}\right)$$

whence $A_1 + B_2$ is increased by

$$\frac{3}{4}\frac{\mu}{r^3}\left(\frac{C-A}{B}-\frac{C-B}{A}\right)^1\frac{A_1-B_2}{(n-n^1)\,n^1}$$

where A_1 and B_2 on the right-hand side stand for the first approximate values of A_1 and B_2.

Similarly $A_1 - B_2$ is increased by

$$\frac{3}{4}\frac{\mu}{r^3}\left(\frac{C-A}{B}-\frac{C-B}{A}\right)\frac{A_2+B_1}{(n-n_1)\,n^1}$$

The increment, therefore, of N introduced by these, will be

$$\frac{1}{C}\frac{\overline{\mu^2}}{r^3}\frac{15}{9}\frac{3}{4}\left(\frac{C-A}{B}-\frac{C-B}{A}\right)\left\{\overline{x_1y_1z_1}\,\frac{A_1-B_2}{(2n-n^1)(n-n^1)\,n^1}\right.$$

$$\left.-\frac{1}{2}\left(\overline{x_1^2}-\overline{y_1^2}\right)\frac{A_1+B_1}{(2n-n_1)(n-n_1)\,n^1}\right\}$$

which, if we put for A_1, &c., their first approximate values, and reject quantities depending upon higher powers of $B - A$ becomes

$$\frac{B-A}{ABC}\frac{135}{32}\left(\frac{\mu}{r^3}\right)^3\frac{1}{(2n-n^1)\,n^1}\frac{1}{D}\left(\frac{3\left(\overline{x_1^2z_1}+\overline{y_1^2z_1}\right)-\overline{z_1^3}}{r^2}\right)\overline{x_1y_1z_1}. \qquad (B)$$

It now remains to take account of the variations of ϕ produced by those of ω_3,

$$\left(\text{since } \frac{d\phi}{dt} = \omega_3, \text{ \&c.}\right)$$

But with regard to these, it is easily seen that however they modify the preceding results, they cannot destroy them; and for this simple reason, that every term in ω_3 which will give rise to such terms will be some quantity divided by C; and this, again, when substituted in the differential equations for ω_3 will produce a quantity divided by C^2, which therefore cannot destroy the terms above found. To proceed, then, we have in the first place,

$$\frac{d\omega_3}{dt} + \&c. = -\frac{B-A}{C}\,\frac{3}{8}\,\frac{\mu}{r^3}\,\overline{1+\cos\iota}^{\,2}\,\sin(\phi - 2\theta),$$

which will give

$$\omega_3 = n + \frac{B-A}{C}\,\frac{3}{16}\,\frac{\mu}{r^3}\,\overline{1+\cos\iota}^{\,2}\,\sin\overline{2n-2n^1}$$

and hence

$$\phi = nt + \frac{B-A}{C}\,\frac{\mu}{r^3}\,\overline{1+\cos\iota}^{\,2}\,\sin\overline{2n-2n^1} = nt + P\sin\overline{2n-2n^1}$$

suppose,

$$\therefore \cos\phi = \cos n + \tfrac{1}{2}P\cos\overline{3n-2n^1} - \tfrac{1}{2}P\cos\overline{n-2n^1}$$

multiply this by

$$\omega_1 = A_1\cos\overline{n-n^1}, \&c.,$$

and we have

$$\omega_1\cos\phi = \ldots + \tfrac{1}{4}A_1P\cos\overline{2n-n^1} - \tfrac{1}{4}A_1P\cos\overline{3n-2n^1} + \&c.$$

with terms of the same kind for $\omega_2\sin\phi$, arising from $B_2\sin\overline{n-n^1}$

$$\therefore \omega_1\cos\phi - \omega\sin\phi = \tfrac{1}{4}\,\overline{A_1-B_2}\,P\,(\cos\overline{2n-n^1} - \cos\overline{2n-3n^1})$$

$$\therefore \iota = \iota_1 + \tfrac{1}{4}A_1 - B_2P\left(\frac{\sin\overline{2n-n^1}}{2n-n^1} - \frac{\sin\overline{3n-2n^1}}{2n-3n^1}\right)$$

(the terms of the form $\overline{A_2+B}\cos\overline{2n-n^1}$, and may be rejected; because A_2+B_1 contains $B-A$ for a factor; and as they are multiplied by P, which contains the same factor, they would produce terms multiplied by $\overline{B-A}$. Similarly,

$$\phi = \frac{\cos\iota}{\sin\iota}\,\tfrac{1}{4}\,P\,\overline{A_1-B_2}\left(\frac{\cos\overline{2n-n^1}}{2n-n^1} + \frac{\cos\overline{2n-3n^1}}{2n-3n^1}\right.$$

with corresponding quantities for ψ.

There is another way, also, in which terms will arise which must be taken into account, as follows:—The term $P\sin\overline{2n-n^1}$ when introduced into the value of ϕ in the equations for ω_1 and ω_2 will augment the value of A_1 so as to make it become $A_1(1+\tfrac{1}{2}P)$. Similarly, B_2 will become $B_2(1-\tfrac{1}{2}P)$, &c., and hence the term in

$$i\,\tfrac{1}{2}\,\frac{A_1+B_2}{2n-n^1}\sin\overline{2n-n^1}$$

is augmented by

$$\tfrac{1}{2}\frac{A_1 - B_1}{2n - n^1}\sin\overline{2n - n^1};$$

and the terms in ϕ and ψ are augmented in the same manner. Adding, then, these additional increments to those just found, we shall have for the whole increments depending upon P,

$$\iota = \iota + \tfrac{1}{2}\frac{(A_1 - B_2)}{2n - n^1}P\cos(2n - n)\tfrac{1}{4}\frac{(A_1 - B_2)}{2n - n^1}P\cos\overline{2n - 3n^1}$$

$$\phi = nt = \frac{\cos\iota}{\sin\iota}\tfrac{1}{2}\frac{A_1 - B_2}{2n - n^1}P\sin\overline{2n - n^1} + \frac{\cos\iota}{\sin\iota}\tfrac{1}{4}\frac{A_1 - B_2}{2n - 3n^1}P\sin\overline{2n - 3n^1})$$
$$+ P\sin\overline{2n - 2n^1}$$

$$\psi = \frac{1}{\sin\iota}\tfrac{1}{2}\frac{A_1 - B_2}{2n - n^1}P\sin\overline{2n - 2n^1} + \frac{1}{\sin\iota}\tfrac{1}{4}\frac{A_1 - B_2}{2n - 3n^1}P\sin\overline{2n - 3n^1}$$

These terms must be added to those given in ι, &c., by the first approximation, and then substitute for i, &c., in N. Also it must be remembered that corresponding to the term $\sin i\sin\overline{2\phi - \theta}$, which arose from the multiplication together of $\sin\iota\sin\theta\sin\overline{2\phi - 2\theta}$, there will be another term, $\sin\iota\sin(2\phi - 3\theta)$. Making the substitutions, therefore, we shall have

$$\sin\iota\,(\sin\overline{2\phi - \theta} - \sin\overline{2\phi - 3\theta}) = \left(\frac{1}{2}\frac{1}{2n - n^1} + \frac{1}{4}\frac{1}{2n - 3n^1}\right.$$
$$\left. - \frac{1}{n^1}\right)P\,\overline{A_1 - B_2}$$

which will therefore give the term in N

$$\frac{15}{2}\left(\frac{1}{2}\frac{1}{2n - n^1} + \frac{1}{4}\frac{1}{2n - 3n^1} - \frac{1}{n^1}\right)P.\overline{A_1 - B_2}$$

the last part of this, however, namely, that multiplied by $-\frac{1}{n^1}$ will vanish; for the term $\sin i\sin\theta\sin(2\theta - 2\theta^1)$, will produce in term in ϕ of the form $Q\,\overline{A_1 - B_2}\cos\overline{2n - n^1}$, which, when introduced into the first part of N, viz.

$$-\frac{B - A}{C}\sin\overline{2\theta - 2\theta^1}$$

it is easily seen, will destroy the term spoken of. If, then, we cancel this term, and substitute for P, $\overline{A_1 - B_2}$, their values, we shall have, observing that such a quantity as $\frac{A}{C}$ may be replaced very nearly by unity, and rejecting any terms multiplied by products of $\overline{B - A}$ and $\overline{C - A}$, &c., for this part of N

$$\left(\frac{\mu}{r^1}\right)^3 \frac{15}{8} \frac{3}{8} \frac{3}{2} \frac{B - A}{ABC} \left(\frac{2}{2n - n^1} + \frac{1}{2n - 3n^1}\right)$$

$$\left(\frac{3\,(\overline{x^2z} + \overline{y_1{}^2z}) - \overline{z_1{}^3}}{r^2\,(n - n^1)}\right) \overline{x_1y_1z_1}\, \frac{1}{D} \cdot \cdot \cdot \cdot \; C$$

The terms found are the only ones depending upon the same combination of $\overline{x_1{}^2z_1}$, &c., and having the same divisor D_1, and not multiplied by sin ${}^2i_1$, or higher powers of $B - A$.

There is one other term, however, it will be advisable to take into account. It arises thus: the term P sin $\overline{2n - 2n^1}$, when introduced into cos $\overline{\theta - \theta^1}$, &c., in the equations for ω_1 and ω_2, produces not only terms having $n - n^1$, but also terms having $3n - 3n^1$ for their arguments; so that ω_1, &c., will contain terms of the form H cos $\overline{3n - 3n^1}$, &c. Now, these terms will obviously contain the same combination of $x_1{}^2z_1$, &c., in their coefficients, that A_1, &c., do; and for this reason they had better be retained. Their divisors, however, will be deficient; but if we neglect quantities depending upon products of

$$\frac{C - B}{A} \; \frac{C - A}{B}$$

it is plain that the divisor of the latter may be put equal to $\frac{1}{3}$ of the divisor of the former; in fact, H will become $\frac{1}{6}\, A_1P$.

In like manner if ω_2 contains the term K sin $\overline{3n - 3n^1}$, we shall have $K = \frac{1}{6}\, B_2P$. The effect of these terms will be to introduce into ι the term

$$\frac{1}{12} \frac{A_1 - B_2}{2n - 3n^1}\, P \sin \overline{2n - 3n^1},$$

with similar terms in ϕ and ψ; and these, when introduced into the term $-$ sin ι sin $\overline{2\phi - 3\theta}$ will give

$$-\frac{15}{2} \frac{1}{12} \frac{1}{2n - 3n^1}\, P\, \overline{A_1 - B_2},$$

that is, they will reduce the term multiplied by

$$\frac{1}{2n - 3n^1}$$

in equation C to $\frac{4}{5}$rds the value it has there.

Suppose, now, we make this reduction, and collect the results, as given by equations (A_1), (B_1), and (C), and we have, finally,

$$\frac{d\omega_3}{dt} = \frac{135}{32}\left(\frac{\mu}{r^3}\right)^3 \frac{B-A}{ABC}\,\frac{1}{D}\,\frac{1}{n-n^1}\left(\frac{1}{n^1}+\frac{1}{6\,.\,2n-3n^1}\right)\frac{1}{r^2}$$
$$\Sigma\left(m(3(x_1^2z^1+y_1^2z_1)-z^3)\right)\Sigma m\,(x_1y_1z_1).$$

It contains, therefore, a constant term; and hence ω_3 will contain a new periodic term multiplied by the time.

Corresponding also to equation (A), there will be a term arising from the second term in the first part of N, viz., $\sin^2\iota \sin 2\phi$, and in (B) a term arising from $\cos\phi$ in the equation for ω_1, &c., as has been before mentioned: but these destroy each other.

There will also be constant terms depending upon the arguments

$$\theta^1,\ \phi - 2\theta^1, \text{ and } 2\phi - \theta^1,$$

which arise in the same way as those given above; that is, in every case the term arising from $\omega_1\omega_2$ on the left side is identically destroyed by a term on the other side, while on the other hand the first term of the function N will produce, just as above, a term which is not destroyed, and another arising from the substitution of the next approximate values of ω_1, &c., in the second part of N; also in all cases the constant term produced by the function $\sin^2 i \sin 2\phi$ in the first part of N is identically destroyed. The constant terms, however, in all these latter cases, are multiplied by $\sin^2\iota$, and depend upon quite different constant qualities. The one which has been examined appears to be the most important, and is the only one which would exist independently of $\sin i$; that is, when the plane of the equator does not differ sensibly from that of the orbit. The rest it is needless to say more about at present; and it only remains to discuss that already found.

Discussion of the foregoing Results.

(1.) Application to the Earth.

The principal object there is to ascertain whether sufficient change has been produced in the motion of the earth to be perceptible by the observations of the last 2000 years. In the term $\frac{\mu}{r^3}$ which occurs, put $r = a$, and multiply numerator and denominator by $\mu + \mu_1$ where μ_1 is the mass of the earth, it becomes

$$\frac{\mu}{\mu+\mu_1}\,\frac{\mu+\mu_1}{a^3}, \text{ or, putting } \frac{\mu+\mu_1}{a^3} = n'^2,$$

we have

$$\left(\frac{\mu}{r^3}\right)^3 = \left(\frac{\mu}{\mu+\mu_1}\right)^3 n'^6.$$

And for the moon, disturbed by the earth, we should have for the corresponding quantity

$$\left(\frac{\mu_1}{\mu+\mu_1}\right)^3 n'^6.$$

Also $D = (n - n^1)^2$, nearly.

The expression for $\frac{d\omega_3}{dt}$, may therefore be written

$$\frac{B-A}{C}\,\frac{135}{32}\left(\frac{\mu}{\mu+\mu_1}\right)^3 \frac{n'^6}{(n-n^1)^3}\left(\frac{1}{n^1}+\frac{1}{6\,(2n-3n^1)}\right)\frac{a_1^{\,2}}{a^2}$$

$$\frac{\Sigma\,(m\,(3(x^2z+y^2z)-z^3))}{Aa_1}\;\frac{\Sigma\,(m\,x_1y_1z_1)}{Ba_1}$$

Where a_1 is the radius of the earth, a the radius of the orbit, to estimate the numerical value of this, we have

$$\left(\frac{\mu}{\mu+\mu_1}\right)^3 = \left(\frac{1}{80}\right)^3$$

$$\frac{n'^6}{(n-n^1)^3}\left(\frac{1}{n^1}+\frac{1}{6\;2n-3n^1}\right) = n'^2\left(\frac{1}{30}\right) \text{ nearly, since} \left(\frac{n^1}{n} = \frac{1}{30};\right.$$

also $\frac{a_1^{\,2}}{a^2} = \frac{1}{60^2}$; as to the quantity $\frac{\Sigma m\,(3(x^2z+y^2z)-z^3))}{Aa_1}$

it appears to be identical with one which occurs in the "Lunar Theory," and which has been estimated by M. Bessel from a great number of observations on the pendulum at about $\frac{1}{3000}$ (see M. Pontecoulant, "Système du Monde," vol. iv., p. 497). And we cannot, without making unwarrantable suppositions, attribute a greater value than this either to $\frac{B-A}{C}$, or to the remaining quantity. Suppose, to fix the ideas, that we were to put each of them at about $\frac{1}{3000}$. Also, the multiplier

$$\frac{n'^2}{30^3} \text{ may be written } \frac{n^1}{n}\,\frac{n^1 n}{30^3}, \text{ or } \frac{n^1 n}{30^4};$$

and to find the *annual* variations we must take n^1 to represent the mean motion in a year, i. e. about 17.000000″, which, expressed in linear measure, may be put at about 85, so that we have for the annual variation of n

$$\frac{135}{32}\,\frac{1}{80^3}\,\frac{85}{30^4}\,\frac{1}{60^2}\,\frac{1}{3000^3};$$

and the change in 2500 years will be thus multiplied by 2500, i. e. about

$$\frac{1}{70}\,\frac{1}{(1000000)^3}\,n.$$

Now, to produce an effect such as is required to account for the acceleration of the moon, i. e. about $^2z_1$, it has been shown in the "Connaissance du Temps," for 1800, that a variation in the length of the day of about a ten-millionth of a second, or, roughly, of about a million millionth part of the entire length would suffice; whereas, the value above given only amounts to about a seventy millionth part of this. And in like manner we shall find that the other terms mentioned, viz. those depending upon the argument θ^1, will only produce quantities which may be put down as quantities of the same order, and apparently not more important than the above. The above effect is wholly due to the moon; that owing to the sun will be still smaller. Also, the earth has been treated as a solid body, no account being taken of its being partially covered with fluid. But Mr. Airy, I believe, has shown that the effect of the friction of the tidal wave in altering the rotation is quite insensible; so that we may conclude that no effect sufficient to be perceptible in 2000 years has been produced by the action of gravity; and hence direct perturbation of the rotatory motion is wholly inadequate to explain the phenomena mentioned.

A possible explanation, however, may be given by supposing that the slight alterations in figure which have been going on from geological or other causes may have been of a kind that shall suffice to cause such an alteration as would be required. If ω is the angular velocity, it is well known that

$$\omega^2 = \frac{c^2}{\Sigma mr^2}$$

c being a constant, m and r the mass and distance from the centre of any particle. Now, the mass remaining constant, it will be seen that a variation in length of the radius of gyration amounting to about a *three thousandth part of an inch* would be sufficient to produce the requisite effect. It is no violent supposition to suppose that such a change may have happened: perhaps the real difficulty would be, on the other hand, to suppose that a body with a radius of 4000 miles had remained so constant. But, on the other hand, the variation ought to be in *increase*, not a diminution in the radius; and there is nothing to show that any changes that have taken place have tended in this direction rather than the other, further than this:—that any departure from the circular form in the sections parallel to the Equator tends to increase the radius of gyration; so that if the general effect of geological changes has been to increase the inequalities of the surface, their effect would also have been slightly to lengthen the day, and hence, also, to cause an apparent acceleration of the moon.

(2.) Application to the Moon.

The chief question of interest which here occurs is: "what are the conditions of stable rotation?" or, in other words, what must be the relation between n and n^1 in order that there may be no permanent variation in the rotation? To answer this, it will be necessary to put down the general form which the equation for $\frac{d\omega_3}{dt}$ will assume after the approximate values, including those arising from perturbation, have been substituted for ι, ϕ, and θ^1 in the general equation. When this has been done, its general form will be

$$\frac{d\omega_3}{dt} = N_0 + A_1 \cos(\overline{n - n^1}\, t + \epsilon) + B_1 \sin(\overline{n - n^1}\, t + \epsilon)$$

$$+ A_2 \cos(\overline{2n - 2n}\, t + 2\epsilon) + B_2 (\sin \overline{2n - 2n}\, t + 2\epsilon) +, \text{\&c.}$$

$$+ P\, \overline{\sin \iota}^{p-q} \cos(\overline{p - q}\, t + q\epsilon) + \theta\, \overline{\sin \iota_1}^{p-q} \sin(\overline{p - q}\, t - q\epsilon) +, \text{\&c.}$$

where N_0 is the constant part, and where A_1, &c., do not contain $\sin \iota_2$ as a multiplier, and where the last line represents the general form of those terms which are multiplied by $\sin \iota_1$, of which the terms in the expansion of N which have been used are an example, viz.

$$\frac{\mu'}{r^3} \sin \iota \frac{15}{2} \cos(2n - n^1 t + \epsilon), \text{ \&c.}$$

Let us omit for a moment the periodic terms, and consider the value of N_0, or rather that part of it found above, and which will be the only part when $\sin \iota_1 = 0$. This term may be put under the form

$$\frac{H}{D} \frac{1}{n - n^1} \left(\frac{1}{n^1} + \frac{1}{6\; 2n - 3n^1} \right)$$

Now this changes its sign, first, when the latter factor does, i. e. when $12n - 7n_1$ becomes 0; that is, when n is something less than $\frac{3}{2}n^1$. This, therefore, would, as far as this term is concerned, be one condition under which the rotation would be stable; but this would be no more than an approximate value, because it does not take account of terms multiplied by $\sin {}^2\iota$, which, for the moon, though small, is not 0. Again, it changes its sign when $n - n^1$ does; that is, supposing the coefficient of $\frac{1}{n - n^1}$ to be positive, it would cause an acceleration or retardation according as n was greater or less than n^1; but in order that there might be stable rotation when $n = n^1$, it is evident that $\frac{d\omega_2}{dt}$ ought to change its sign by passing through *zero*, not by passing through infinity, as it appears at first sight to do when $n - n^1 = 0$. Let us examine what the true value is under such circumstances. It is quite evident that it must be either 0 or infinite, since it is only by passing through one or other of these that it can change its sign. To see which it really is, it will be necessary to look back to the process by which the function containing it was formed; and if we do so, we shall see that the term is in reality only the first of a series, consisting of odd numbers of $\overline{n - n^1}$, having its signs alternately positive and negative; so that although, if we were to take only the first turn, it would certainly be infinite when $n - n^1 = 0$; yet it does not follow that the entire series is so. On the other hand, it is quite evident that it is not; for it arises from the multiplication together of such terms as sin ι and sin n. Now, whatever be the value of ι or x, sin ι and x are never greater than unity, and hence their product must consist of terms whose sum cannot be greater than unity. In expanding sin x, &c., we only took the terms of lowest dimension that occurred, that is, we put

$$\iota = \iota_1 + a, \quad \text{or } \sin \iota = \sin (\iota_1 + a)$$

$$= \sin \iota \left\{1 - \tfrac{1}{2}a^2 + \frac{1}{4}a^4 + \right\} + \cos \iota \left(a - \frac{a^3}{3} - \text{\&c.}\right)$$

The higher powers of the quantities represented by α were rejected; if they had been retained, we should have had a series such as that mentioned. If, then, the nonperiodic term cannot become infinite, and yet changes its sign when n becomes equal to n^1, it is plain that for such a value it must disappear, and that as far as this term only is concerned, $n - n^1 = 0$ will be approximately a condition of stable rotation. If, however, we had taken into account terms depending upon the arguments θ^1 $2\phi - \theta^1$, &c., it will be found that we should have introduced into the constant terms quantities multiplied by $\sin^2 \iota$, and which do not change their sign when $n = n^1$; so that the entire value of N_0 will be a quantity which does not change its sign, and cannot become 0 when $n = n^1$; and the relation between n and n^1 thus obtained by equating N_0 to 0 will be the relation which ought to be used instead of $n = n^1$; but it would appear also that in *addition* to this relation, the conditions will also be very approximately satisfied by $n = n^1$. To show this, let $N_{n=n^1}$ be the value which N_0 assumes when $n = n^1$.

Then, the equation for $\frac{d\omega_3}{dt}$ becomes

$$\frac{d\omega_3}{dt} = N_{n=n^1} + A_1 \cos \epsilon + B_1 \sin \epsilon + A_2 \cos 2\epsilon + B_2 \sin 2\epsilon.$$

A_2 may be rejected. Now, if we put this $= 0$ it will give us a value by which ϵ may be determined so as to satisfy the equation $\frac{d\omega_3}{dt} = 0$ when $n = n^1$; for, for no value of ϵ can sin ϵ and cos ϵ be simultaneously equal to 0; and since $N_{n=n^1}$ is very small compared with A_1 B_1, and especially B_2, it is evident that a possible value of ϵ may be found to satisfy the equation

$$N_{n=n} + A_1 \cos \epsilon + \text{\&c.} = 0.$$

In other words, it must have a principal axis inclined at a particular angle to the radius vector of the disturbing body; and this angle, though it appears to be small, cannot be 0. And these appear to be the only conditions when the mean value of ι is nothing; but in other cases there will be other conditions, which will be seen thus:—Let $N_{(p-q)}$ represent the value which N_0 takes when the relation between n and n^1 is such that $pn = qn^1$. Then, when this is the case, the equation for ω_3 becomes

$$\frac{d\omega_3}{dt} = N_{(p-q)} + P\,\overline{\sin\iota_1}^{\,p-q}\sin(p\epsilon) + \theta\,\overline{\sin\iota_1}^{\,p-q}\cos(p\epsilon) + \text{periodic terms.}$$

Now, the terms $\sin(p\epsilon)$ not containing the time, are constant quantities, and to find, when $\frac{d\omega_3}{dt}$ is 0, we must equate the three terms given above to zero, and this will give us an equation for determining what ϵ must be in order that the body may revolve permanently with the relation $pn = qn^1$. That such a condition may be possible, will, of course, imply that the coefficients, &c., in the above equation, have such a value as to give values of $\sin p\epsilon$ and $\cos p\epsilon$ not greater than unity. And it is evident, in order that such a requisition may be fulfilled, the coefficient of either of them cannot be small with respect to $N_{\overline{p-q}}$. And this will show, that though there are several such relations, there can only be a limited number; for, as the quantities p and q become large, so also does the power of $\frac{a_1}{a}$ or $\frac{1}{60}$, which multiplies such terms, become large; and hence for large values of p and q, the coefficients in question rapidly diminish, the more so when ι is small, and there will not be many of them which are larger than N_{pn-qn}, and consequently not many different conditions of stable rotation. *How* many there are it is, of course, impossible to say without more knowledge than we have, or ever can have, of the numerical values of the various quantities concerned.

We may conclude, then, certainly, that there will always be either an acceleration or retardation of a body revolving freely about a fixed point, and acted on by a disturbing force moving round it, except when certain given relations exist between n and n^1; but *which* of the two it will be it is quite impossible to say, without knowing more about the form of the revolving body than we do of the moon; for without such knowledge we cannot determine the algebraic signs of the various coefficients. One thing, however, appears highly probable, and it is this: that if the conditions of equable rotation can be satisfied by values of n not *very* much greater than n^1 such as $n = \frac{3}{2}n^1$, or, on the other hand, by values not very much less than n^1, such as $n = \frac{2}{3}n^1$, if its rotation had ever been very much greater, or very much smaller than it is, it would seem that the change ought to have ceased when it came to a position of equable rotation, such as either of the former, without further diminishing or increasing till it became equal to n^1; and from hence it would appear that its rotation can never have been very different from what we actually observe it to be.